Bounds on positive solutions of polynomial systems

Master's Thesis

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Motivation

In the univariate case, we already know how many complex roots a polynomial has.

Fundamental Theorem of Algebra

A polynomial $f \in \mathbb{R}[x]$ of degree n has precisely n roots in $\mathbb{C}$ counted with multiplicity.

But what about real roots? Determining how many real solutions can be found for a system of polynomials is important in many applications. In particular, it is often important to determine, whether there exists any positive real roots.

The Univariate Case

While the multivariate case is still an open problem, the univariate case has been solved. In fact, we know a lot about it, and there are many sufficient conditions for real rootedness, as well as several different bounds on the number of real roots. The oldest of which is probably due to Descartes.

Descartes' rule of signs [Sot11]

Let $f = \sum_{i=1}^{n} c_i x^{a_i}$ be a real polynomial, where $c_i \neq 0$ and $a_1 < \cdots < a_n$. Let r be the number of positive roots of f counted with multiplicity. We have that

$$\operatorname{sgnvar}(c_1, \dots, c_n) \geq r,$$

and the difference is even.

Example: Consider the polynomial $f = x^5 + 4x^4 + 2x^3 - 2x^2 + x - 6$. As $\operatorname{sgnvar}(-6, 1, -2, 2, 4, 1) = 3$, it follows that f has either one or three positive real roots. In particular, we are guaranteed at least one positive real root.

A condition for real-rootedness

While Descartes' rule of signs gives a bound on the number of positive solutions, it does not give any condition for a polynomial to have all roots real. The following result provides exactly that.

Theorem [Kur92]

Let $f = \sum_{i=0}^{n} c_i x^i$ be a polynomial of degree $n \geq 2$ with positive coefficients. If

$$c_i^2 - 4c_{i-1}c_{i+1} > 0, \quad i = 1, 2, \dots, n-1, \tag{1}$$

then all the roots of f are real and distinct.

For $n = 2$, eq. (1) is simply the usual discriminant condition. For $n = 3$, the inequalities from the theorem are $a_1^2 - 4a_2a_0 > 0$, and $a_2^2 - 4a_3a_1 > 0$.

My Experiment

- **Question:** If we are given a real-rooted polynomial with positive coefficients, then how "often" will the condition from eq. (1) be satisfied?
- **Experiment:** Generate polynomials with real roots in some interval and test whether they satisfy eq. (1). Results are listed as ratios between the number of instances satisfying the conditions and the sample size.

Results are listed in Table 1. The results raise some questions:

- Why does the ratio increase, when the interval becomes bigger?
- What happens in the interval $(-1, 0)$?

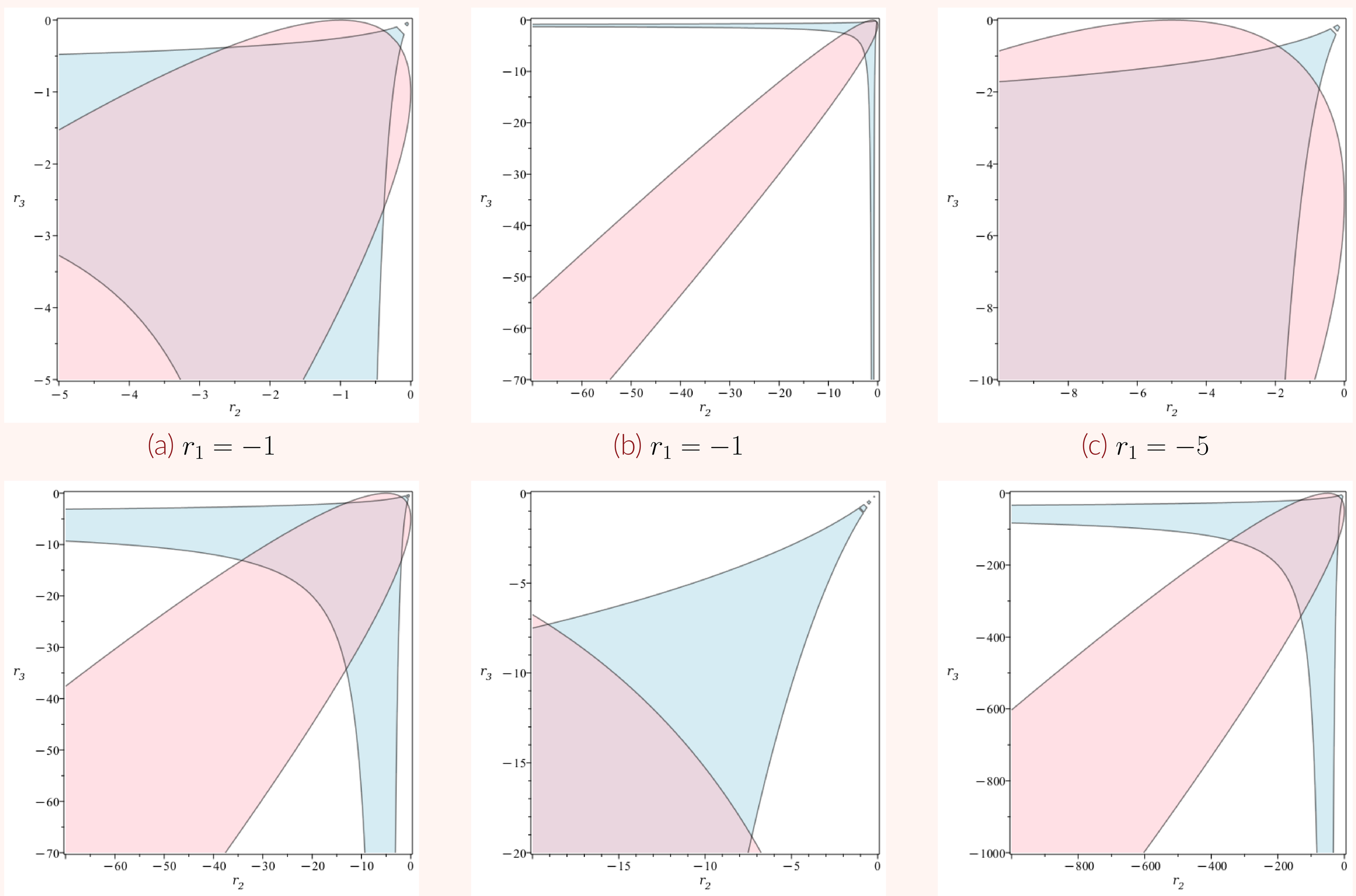
To try to answer these questions, we consider an example. Let $f(x) = (x - r_1)(x - r_2)(x - r_3)$. Then eq. (1) corresponds to

$$\begin{aligned} 0 &< r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 - 2r_1^2r_2r_3 - 2r_1r_2^2r_3 - 2r_1r_2r_3^2, \\ 0 &< r_1^2 + r_2^2 + r_3^2 - 2r_1r_2 - 2r_1r_3 - 2r_2r_3. \end{aligned} \tag{2}$$

See Figure 1 for a plot of these inequalities.

Interval	Degree 3	Degree 4	Degree 5	Degree 6
$(-100, 0)$	0.11938581	0.00486293	$5.144 \times e^{-5}$	$8.0 \times e^{-8}$
$(-1000, 0)$	0.11937622	0.00485887	$5.231 \times e^{-5}$	$1.6 \times e^{-7}$
$(-10000, 0)$	0.11934174	0.00487097	$5.08 \times e^{-5}$	$1.3 \times e^{-7}$
$(-100000, 0)$	0.11938009	0.00485577	$5.278 \times e^{-5}$	$1.6 \times e^{-7}$
$(-1000000, 0)$	0.11938332	0.00487046	$5.13 \times e^{-5}$	$1.2 \times e^{-7}$
$(-100, -1)$	0.09765286	0.00157997	0.0	0.0
$(-1000, -1)$	0.11685948	0.00442414	$3.061 \times e^{-5}$	$3.0 \times e^{-8}$
$(-10000, -1)$	0.11908152	0.00481732	$4.943 \times e^{-5}$	$9.0 \times e^{-8}$
$(-100000, -1)$	0.1193003	0.00485834	$5.111 \times e^{-5}$	$1.3 \times e^{-7}$
$(-1000000, -1)$	0.11941458	0.00486441	$5.016 \times e^{-5}$	$1.4 \times e^{-7}$

Table 1. All experiments are run with 100,000,000 samples.



(a) $r_1 = -1$ (b) $r_1 = -1$ (c) $r_1 = -5$

(d) $r_1 = -5$ (e) $r_1 = -50$ (f) $r_1 = -50$

Figure 1. Fixing r_1 and plotting eq. (2). The white area is where both inequalities hold.

Motivating the multivariate case

Again, we want a bound on the number of (positive) real roots. We would like a bound, which does not depend on the degrees, nor on the volume of the Newton polytope. An important step in this direction is Khovanskii's revolutionary fewnomial bound.

Khovanskii's Fewnomial Bound, [Sot11]

A system of n polynomials in n variables involving $n+l+1$ distinct monomials has strictly less than $2^{\binom{n+l}{2}}(n+1)^{n+l}$ nondegenerate positive solutions.

Descartes' rule of signs for circuits

Fix an ordered set of exponent vectors $\mathcal{A} = \{a_0, \dots, a_{n+1}\} \subset \mathbb{Z}^n$. Assume $\mathcal{A} \in \mathbb{Z}^n$ to be a circuit. Let $C \in \mathbb{R}^{n \times (n+2)}$ be a coefficient matrix. For any such matrix, we denote by

$$f_i(x) = \sum_{j=0}^{n+1} c_{i,j} x^{a_j} = 0, \quad i = 1, \dots, n. \tag{3}$$

the corresponding fewnomial system in n variables $x = (x_1, \dots, x_n)$ with support $\mathcal{A}$.

Let $n_{\mathcal{A}}(C)$ be the number of positive real solutions to the system in eq. (3). Define the matrix

$$A = \begin{pmatrix} 1 & \dots & 1 \\ a_0 & \dots & a_{n+1} \end{pmatrix}.$$

We assume that C has rank n and that $0 \in \operatorname{pos}(C_0, \dots, C_{n+1})$. Let $P \in \mathbb{R}^{(n+2)\times 2}$ be a Gale dual of C (i.e., P has columns $\{P_0, \dots, P_{n+1}\}$, which are a basis of the kernel of C).

Definition

Define an equivalence relation $\sim$ on $[n+2]$ to be

$$i \sim j \iff \det(P_i, P_j) = 0.$$

Let k be the number of equivalence classes ($1 \leq k \leq n+2$). We denote by

$$[n+2]/\sim\, = \{K_0, \dots, K_{k-1}\}$$

the set of equivalence classes. The ordering on $[n+2]/\sim$ is the canonical ordering of the equivalence classes.

Pick some ordering within each equivalence class and combine it with the canonical ordering of the equivalence classes to get a permutation $\sigma \in S_{n+2}$, such that there exists $\varepsilon = \pm 1$ satisfying

$$\varepsilon \det(P_{\sigma(i)}, P_{\sigma(j)}) \geq 0, \quad \text{for all } i < j.$$

Here, ε depends on the orientation chosen, when ordering the equivalence classes. Let $K \subset [n+2]$ be a set of representatives of the classes in $[n+2]/\sim$. Then σ induces a bijection $\tau\colon [k] \to K$. It follows that $\epsilon \det(P_{\tau(i)}, P_{\tau(j)}) > 0$.

We call σ and *ordering* for C and τ a *strict ordering* for C.

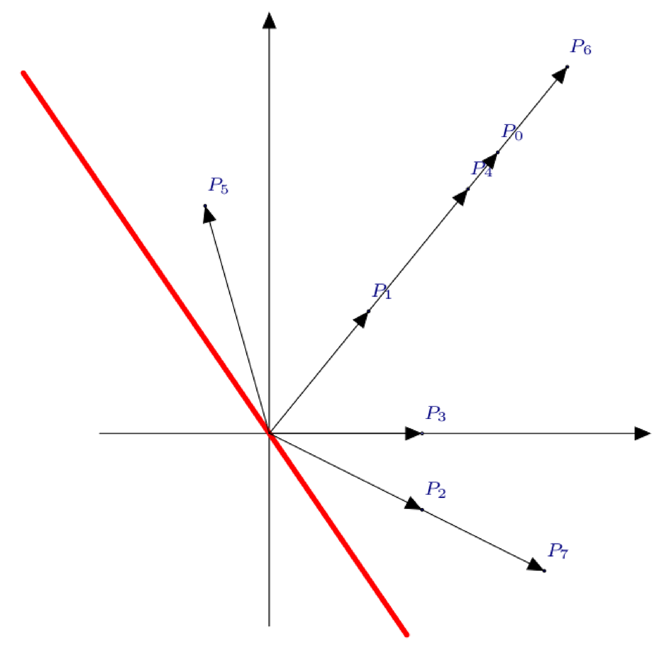


Figure 2. Example of configuration $\{P_0, \dots, P_6\}$ (from [BDF22, Figure 3]).

Let $B \in \mathbb{Z}^{(n+2)\times 1}$ be any matrix, which is Gale dual to A (i.e., column vector $b = (b_0, \dots, b_{n+1})^T$ of B is a basis of $\ker(A)$). Hence, up to multiplication by some constant,

$$b_j = (-1)^j \det(A(j)).$$

Definition

Let $L_l = K_0 \cup \cdots \cup K_l$ for each $l \in [k]$. Then we can define the terms $\lambda_l = \sum_{j \in K_l} b_j$ and $\mu_l = \sum_{j \in L_l} b_j$, which gives ud the sequences $\lambda = (\lambda_l)_{l \in [k]}$ and $\mu = (\mu_l)_{l \in [k]}$.

Descartes' rule of signs for circuits [BDF22]

If $n_{\mathcal{A}}(C)$ is finite, then $n_{\mathcal{A}}(C) \leq 1 + \operatorname{sgnvar}(\mu)$. In particular, $n_{\mathcal{A}}(C) \leq k - 1 \leq n + 1$.

References

[BDF22] Frédéric Bihan, Alicia Dickenstein, and Jens Forsgård. Optimal Descartes' Rule of Signs for Circuits. 2022.

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